Interlaboratory Study Protocol

Calibration of Optical Systems for Dynamic and Static Deformation Measurement

Version June 2013

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for Dynamic and Static Deformation Measurement

This protocol is based on the "Guidelines for the Calibration of Optical Systems for Dynamic Deformation Measurement" that were prepared as part of the ADVISE project (Grant Agreement SCP7-GA-2008-218595) within the European Commission's Safety and Security by Design Programme (FP7-SST-2007-RTD-1). It has been adapted in the VANESSA Coordination and Support Action (Grant Agreement FP7-NMP-2012-CSA-6).

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Executive Summary

In the last decade there has been a high level innovation in the field of experimental mechanics with advances in both the capabilities of existing techniques such as digital photoelasticity and the development of powerful new techniques such as digital image correlation. The wider structural analysis community has been slow to embrace these developments and continues to almost exclusively utilise computational methods of strain and modal analysis. However, the need for data from experiments to validate numerical models is becoming more important, particularly as new materials and complex structures provide severe challenges to reliable simulations. Calibration procedures are required to provide user and regulator confidence in optical techniques of deformation measurement employed in validation. Certification of calibration requires traceability, which is derived from an unbroken sequence of comparisons to form a chain to the primary or national standard, and in this case the international standard for the metre has been adopted. In addition to the certification process, calibration provides the minimum measurement uncertainty that can be expected from the calibrated instrument and supports refinement of the instrument to optimise its performance.

The purpose of this document is to provide a step-by-step protocol for the use of a Reference Material for the calibration of optical systems for full-field measurement of dynamic strain and, or displacement induced by cyclic and, or static loading, in order to establish traceability to the international primary standard. The Reference Material is in the form of a stepped bar that is clamped rigidly at its thick end to a large, immovable body to create a cantilever, which is subjected to cyclic or static loading. Analytical descriptions of the in-plane strain fields and out-of-plane displacements for the cantilever, as a function of the displacement at its tip, form the basis for comparison to the primary standard for length.

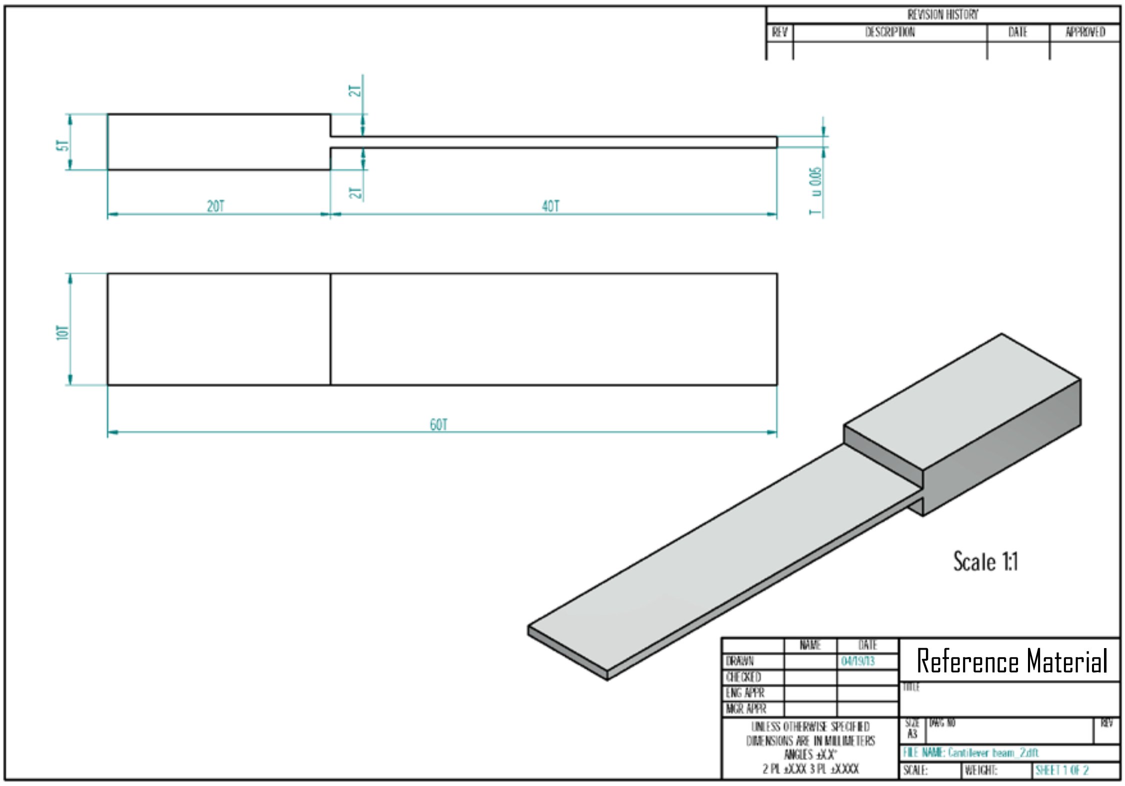

Figure 1: Drawing and three-dimensional rendering of the Reference Material. The thick section is intended to be rigidly clamped to a large, immovable body. Manufacturing guidelines are given in Appendix A.

1. Description of Reference Material

1.1 Design

The design of the Reference Material consists of a simple cantilever in the form of a stepped bar as shown in Figure 1. The cantilever has thickness T, length 40T, and width 10T, with at one end an enlarged portion of thickness 5T and length 20T, giving an overall length of 60T.

The origin of coordinates is at the intersection of the axis of symmetry of the Reference Material with the section change, such that x is measured along the cantilever towards the tip, y across the face of the cantilever, and z from the neutral axis through the thickness.

The enlarged portion is used for clamping the cantilever to a rigid, immovable body. Experiments have shown that the behaviour of the cantilever is independent of the clamping method and force, provided there is no relative movement between the enlarged end of the Reference Material and the rigid, immovable body.

The design is scalable and the dimensions and materials should be chosen so that deformations are comparable to those it is expected to measure with the calibrated measurement system. However, the Reference Material should remain in the elastic loading regime so that the process is reproducible.

1.2 Natural frequencies

The natural frequencies, f_k for the resonant bending modes of the Reference Material can be found analytically as[1]

$$f_k = \frac{\lambda_k^2}{2\pi L^2}\left(\frac{EI}{M}\right)^{1/2} \tag{1}$$

where L is the length of the cantilever, E is the modulus of elasticity, I is the second moment of area and M is the mass of the cantilever, which in this case is $2M_{RM}/7$ where M_{RM} is the mass of the complete Reference Material. The second moment of area is defined as $I=bh^3/12$, which for the Reference Material gives $I=5T^4/6$.

The dimensionless natural frequency parameters, λ_k, are computed from

$$\cos\lambda \cosh\lambda + 1 = 0 \tag{2}$$

The first few values are given in **Table 1**. For k > 4 use $\lambda_k = (2k-1)\frac{\pi}{2}$ and $\phi_k = 1$.

Table 1: Mode shape parameters of a cantilever beam clamped at one end.

λ_1 = 1.875	λ_2 = 4.694	λ_3 = 7.855	λ_4 = 10.996
λ_1^2 = 3.516	λ_2^2 = 22.03	λ_3^2 = 61.70	λ_4^2 = 120.9
ϕ_1 = 0.7340	ϕ_2 = 1.0185	ϕ_3 = 0.9992	ϕ_4 = 1.0000

[1] Blevins, R.D., 2001, *Formulas for natural frequency and mode shape*, Krieger Pub. Co., Malabar, FA.

1.3 Deformation field equation for resonant bending modes

The modal shape for bending mode *k* is given by (Appendix B)

$$w_k\left(\frac{x}{L}\right)=\frac{\delta}{2}\left\{\cosh\lambda_k\frac{L-x}{L}+\cos\lambda_k\frac{L-x}{L}-\phi_k\left(\sinh\lambda_k\frac{L-x}{L}+\sin\lambda_k\frac{L-x}{L}\right)\right\} \quad (3)$$

where w_k is the out-of-plane displacement of the cantilever in bending mode *k*, δ is the tip amplitude, *x* is measured from the clamped end along the cantilever, and ϕ_k is given by

$$\phi_k=\frac{\sinh\lambda_k-\sin\lambda_k}{\cosh\lambda_k+\cos\lambda_k} \quad (4)$$

The direct strain along the cantilever, ε_x is given by

$$\varepsilon_x(x,y)=z\frac{d^2w(x)}{dx^2} \quad (5)$$

where *z* is the distance from the neutral axis through the thickness of the cantilever beam, such that at the top surface *z* = *T*/2. The surface strain is given by

$$\varepsilon_k(x,y)=\delta\frac{T\lambda_k^{\,2}}{4L^2}\left\{\cosh\lambda_k\frac{L-x}{L}-\cos\lambda_k\frac{L-x}{L}-\phi_k\left(\sinh\lambda_k\frac{L-x}{L}-\sin\lambda_k\frac{L-x}{L}\right)\right\} \quad (6)$$

The components of surface strain in the *y*-direction are zero except within a distance equivalent to the width of the cantilever (W=10T) from the clamp.

1.4 Deformation field equation for static load

When the cantilever is deformed by a load *P* acting at its tip, then, in the linear regime, the out-of-plane displacement is given by [2]

$$w(x)=\frac{Px^2}{EI}(3L-x) \quad (7)$$

If the displacement at the tip, $v(L)=\delta$ is known then

$$P=\frac{EI\times\delta}{2L^3} \quad (8)$$

and expression (7) can be rewritten as

$$w(x)=\frac{\delta}{2L^3}\left(3Lx^2-x^3\right) \quad \text{or} \quad w\left(\frac{x}{L}\right)=\frac{\delta}{2}\left(3\left(\frac{x}{L}\right)^2-\left(\frac{x}{L}\right)^3\right) \quad (9)$$

Using Eq. (5), the direct strain along the cantilever, ε_x , is given by

$$\varepsilon_x(x,y)=\delta\frac{3T}{2L^2}\left(1-\frac{x}{L}\right) \quad (10)$$

The component of strain on the surface in the *y*-direction is zero except within a distance equivalent to the width of the cantilever (*W*=10T) from the clamp and location of the applied static load.

[2] Young, W.C., 1989, Roark's formulas for stress & strain, 6th ed., McGraw-Hill Book Co., New York

2. Methodology for use

The size and deflection of the Reference Material should be such that they cover the ranges within those it is intended to make measurements with the calibrated instrument.

A flowchart that outlines the steps described below for performing a calibration is shown in Figure 2 and should be read in conjunction with the following sections.

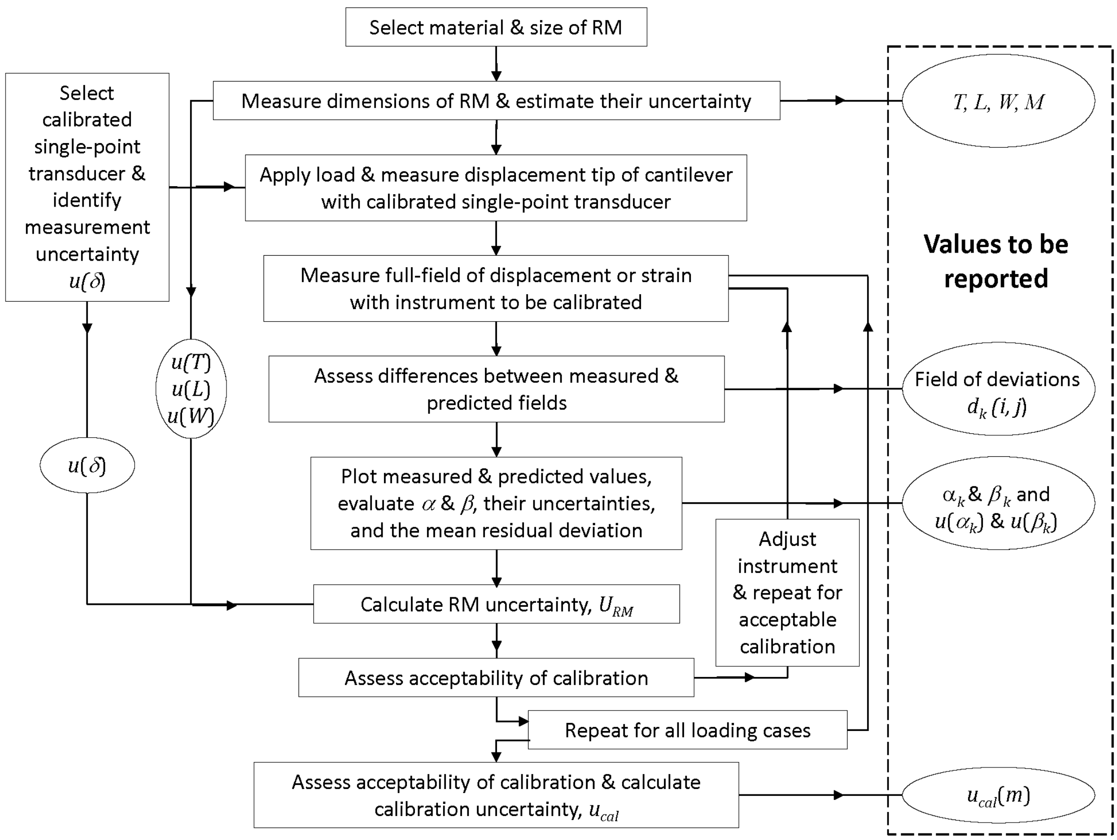

Figure 2: Flow chart for performing a calibration with operations shown as rectangular boxes and quantities as ovals. Quantities that must be reported as part of the calibration are highlighted separately as outputs on the right.

Preparation

It is strongly recommended that the preformat on the opposite page be used.

Note down the Reference Material identification number.

Measure the actual thickness T, length L, width W, and total weight M.

Estimate the uncertainty in these measurements $u(Q)$ [3].

Experimental set-up

Clamp the thick end of the Reference Material rigidly to an immovable body.

Set up the instrument to be calibrated.

- The length L of the cantilever section of the Reference Material must occupy at least 80% of the longer dimension of the detector.
- The cantilever root must be visible as it serves as point of zero deflection.
- The cantilever should be represented by at least 20 × 5 pixels or data points so that the number of data points considered is $N \geq 100$.

Obtain the physical coordinates of the data points, (x_i, y_j).

Note down the parameter settings of the instrument, its identification and other information of operation relevant for calibration (e.g. software parameters, information requested from the Standard Operating Procedure (SOP)).

Install the loading device

(a) For cyclic loading, the Reference Material is excited at a single frequency by any means that is practical and appropriate for the scale being employed and does not involve physical contact with the cantilever. Acoustic loading is recommended. It is recommended the vibration equipment should satisfy the requirements described in ISO 16063-21:2003[4].

(b) For static loading, the tip of the Reference Material should be loaded in contact by any appropriate force introduction device, e.g. a dead-weight suspended from a cross-bar attached to the tip.

Note down the parameter settings of the loading device, its identification and other information relevant for calibration (e.g. software parameters, information requested from the SOP).

Monitor the displacement of the cantilever tip using a calibrated single-point displacement transducer.

- Any calibrated, and hence traceable, single-point displacement transducer which is appropriate to the scale of the deflection may be employed.
- Estimate the uncertainty in the deflection measurements.

Note down the parameter settings of the displacement transducer, its identification and other information relevant for calibration (e.g. software parameters, information requested from the SOP).

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[3] ISO/IEC Guide 98:1995, Guide to the expression of uncertainty in measurements (GUM), Joint Committee for Guides in Metrology 100:2008 (BIPM, Paris 2008).

[4] ISO 16063-21 Methods for the calibration of vibration and shock transducers - part 21: vibration calibration by comparison to a reference transducer, 2003.

Protocol

Preparation	RM Id		
Your comments:	quantity Q	value	uncertainty $u(Q)$
	T =	[mm]	[mm]
	L =	[mm]	[mm]
	W =	[mm]	[mm]
	M =	[kg]	[kg]

Experimental set-up		
Your comments:	**Measuring Instrument Id**	
	Parameters	
	…	
	…	
	…	
	Loading device Id	
	Parameters	
	…	
	…	
	…	
	Deflection Transducer Id	
	Parameters	
	…	
	…	
	…	

Measurement procedure: choose (a) or (b)

(a) Cyclic loading

Sweep the excitation frequency across the appropriate frequency interval.

Obtain the first three resonance frequencies (by watching the displacement signal) of the bending modes. Report the frequencies f_k.

- If the frequency interval over which it is intended to use the calibrated instrument is higher than the third bending resonance frequency, then obtain the resonance frequencies of the successive bending modes until the frequency range is covered.

Obtain the displacement and/or strain map of the cantilever at each resonance frequency.

Obtain simultaneously the transducer signal of the tip deflection and report the amplitude δ.

(b) Static loading

Load the object statically using the load device.

Obtain the displacement and/or strain map of the cantilever.

Obtain simultaneously the transducer signal of the tip deflection and report the amplitude δ.

Protocol: follow (a) or (b) as chosen

(a) Cyclic loading

Measurements: Resonance frequency		**Frequency**
Your comments:	1st : f_1	[Hz]
	2nd : f_2	[Hz]
	3rd : f_3	[Hz]
	…	[Hz]

Measurements: Deformation field	**Data Id.**	**Transducer amplitude**
1st resonance frequency		[mm]
2nd resonance frequency		[mm]
3rd resonance frequency		[mm]
…		[mm]

(b) Static loading

Measurements: Load cases		**Force**
Your comments:	1st load	[N]
	2nd load	[N]
	3rd load	[N]
	…	[N]

Measurements: Deformation field	**Data Id. (file name)**	**Transducer amplitude**
1st load		[mm]
2nd load		[mm]
3rd load		[mm]
…		

Comparison of measured and predicted values

Predict the values m_T corresponding to the data points in (x_i, y_j) for the in-plane strain and, or displacement field of the Reference Material using expressions (3), (6), (9) or (10).

Calculate the difference $d_k(i, j)$ between the predicted values m_T and the measured values m_E of the measurand, *m,* viz. strain or displacement:

$$d_k(i, j) = m_{T_k}(x_i, y_j) - m_{E_k}(x_i, y_j) \tag{11}$$

where k numbers the load cases. The gauge area, A_g, is defined as the entire surface of the cantilever, i.e. $0 < x < L$ and all $-W/2 < y < W/2$.

Plot the difference values $d_k(i, j)$ against $m_{T_k}(x_i, y_j)$ for every load case k.

Perform a linear least-squares fit to the field of deviations, $d_k(i, j)$ to obtain fit-parameters α_k and β_k for each load case which minimize the residual

$$\sum_{i,j} \left[d_k(i, j) - \alpha_k - \beta_k\, m_{T_k}(x_i, y_j)\right]^2 \tag{12}$$

Calculate the fit-parameters, α_k and β_k which must be reported from:

$$\alpha_k = \frac{\sum_{i,j} m_{T_k}^2(x_i, y_j) \sum_{i,j} d_k(i, j) \;-\; \sum_{i,j} m_{T_k}(x_i, y_j) \sum_{i,j} m_{T_k}(x_i, y_j)\, d_k(i, j)}{N \sum_{i,j} m_{T_k}^2(x_i, y_j) \;-\; \left(\sum_{i,j} m_{T_k}(x_i, y_j)\right)^2}$$

$$\beta_k = \frac{N \sum_{i,j} m_{T_k}(x_i, y_j)\, d_k(i, j) \;-\; \sum_{i,j} m_{T_k}(x_i, y_j) \sum_{i,j} d_k(i, j)}{N \sum_{i,j} m_{T_k}^2(x_i, y_j) \;-\; \left(\sum_{i,j} m_{T_k}(x_i, y_j)\right)^2} \tag{13}$$

Evaluate, for each load case k, the mean square residual deviation after the linear fit by using:

$$u^2(d_k) = \frac{1}{N} \sum_{i,j} \left[d_k(i,j)\right]^2 - {\alpha_k}^2 - 2\alpha_k \beta_k \frac{1}{N} \sum_{i,j} m_{T_k}(i, j) - {\beta_k}^2 \frac{1}{N} \sum_{i,j} m_{T_k}^2(i, j) \tag{14}$$

Report the uncertainties of the fit-parameters:

$$u(\alpha_k) = \sqrt{\frac{u^2(d_k)}{N}} \qquad u(\beta_k) = \sqrt{\frac{u^2(d_k)}{\sum_{i,j} m_{T_k}^2(i, j)}} \tag{15}$$

If $|\alpha_k| > 2u(\alpha_k)$, namely α_k is greater than its 'expanded uncertainty', then there is a statistically significant offset in the calibration.

If $|\beta_k| > 2u(\beta_k)$, i.e. β_k is greater than its 'expanded uncertainty', then there is a statistically significant deviation in the calibration. This implies that the difference between the measured and predicted values of strain or displacement is a function of the deformation.

Protocol

Comparison	N =		number of points

Comparison	α_k [mm *or* με]	β_k [mm/mm *or* με/mm]	u(α_k) [mm *or* με]	u(β_k) [mm/mm *or* με/mm]
1st load case				
2nd load case				
3rd load case				
…				

Uncertainty of Comparison		$u(d_k)$ [mm *or* με]
Your comments:	1st load	
	2nd load	
	3rd load	
	…	

Uncertainty of Calibration		$u_{cal}(m)_k$ [mm or με] **see overleaf**
Your comments:	1st load	
	2nd load	
	3rd load	
	…	

Please return the completed proforma and the files of your data fields to

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Empa
Electronics/Metrology/Reliability Laboratory
Ueberlandstrasse 129
CH-8600 Dübendorf

erwin.hack@empa.ch

Calibration uncertainty

Choose from the equations C1 to C4 in Appendix C, depending on the calibration quantity, viz. dynamic or static, displacement or strain.

Calculate the uncertainty of the Reference Material values. The expressions represent the combined standard uncertainty of the Reference Material values, u_{RM}.

Obtain the expanded uncertainty in the Reference Material, $U_{RM} = 2u_{RM}$ for each quantity of interest.

Plot $\pm U_{RM}$ as a function of m_T for each load step or natural frequency, k. The area between these lines defines the uncertainty band arising from the Reference Material (Figure 3).

Plot on the same graph the mean residual deviations, $(\alpha_k + \beta_k\, m_{Tk}) \pm 2u(d_k)$ from equations (13) and (14) where $\pm 2u(d_k)$ indicates the width of the scatter band around the line $\alpha_k + \beta_k x$.

The interpretation of this plot is illustrated in the following two cases:

<u>*Case 1*</u>: When the scatter band $\pm 2u(d_k)$ on $\alpha_k + \beta_k x$ does not overlap the uncertainty band arising from the Reference Material for every value of x then there is a significant deviation and re-calibration should be considered. If the uncertainties in α_k and β_k are large then they cannot be used as correction factors and efforts must be made to improve the quality of the measurement made by the instrument being calibrated.

<u>*Case 2*</u>: When the scatter band $\pm 2u(d_k)$ on $\alpha_k + \beta_k x$ overlaps the uncertainty band arising from the Reference Material for every value of x then no correction to the calibration of the measurement system is needed. In this case the measurement values are consistent with the reference values.

The procedure described above should be repeated for each load case, until the situation in *Case 2* is achieved for every increment.

Then the calibration uncertainty can be approximated using:

$$u_{cal}(m)_k = \sqrt{u^2(d_k) + u_{RM}^2(m)_k} \qquad (16)$$

for a single increment of applied displacement.

Report $u_{cal}(m)_k$ in the proforma overleaf.

<u>*Future experiments:*</u>

The calibrated instrument can be used to measure displacements over the range of deflection and frequencies employed in the calibration process; use beyond this range will require a fresh calibration.

The measurement uncertainty in any future experiment performed with the unchanged measurement system (i.e. no change in the extrinsic and intrinsic parameters of the optical system) is at least as high as the calibration uncertainty.

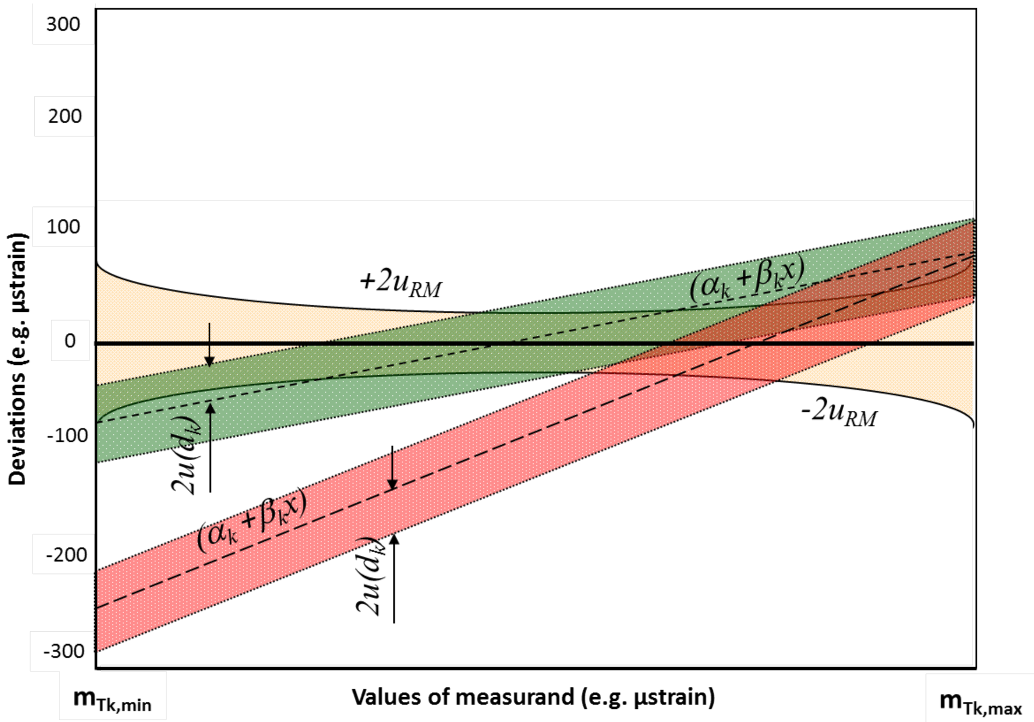

Figure 3: Graph showing the expanded uncertainty of the Reference Material, $\pm U_{RM}$ as a function of the measurand m_T, together with the mean residual deviations $(\alpha_k+\beta_k m_T) \pm 2u(d_k)$ for two cases. In case 1, $\pm U_{RM}$ and $(\alpha_k+\beta_k m_T) \pm 2u(d_k)$ do not overlap for all values of *m,* and so re-calibration following adjustment of the instrument is appropriate. In case 2, $\pm U_{RM}$ overlaps with $(\alpha_k+\beta_k m_T) \pm 2u(d_k)$ for all *m* and so no adjustment or re-calibration is necessary.

found using the procedure described above. Equation (16) corresponds to the calibration uncertainty u_{cal} and the expanded uncertainty U_{cal} is given by:

$$U_{cal} = 2u_{cal} \tag{17}$$

Stating the measurement result and documenting its uncertainty is an essential part of a traceability procedure and hence forms an indispensable part of the process.

Appendix A: Manufacturing guidelines

The design consists of a simple cantilever in the form of a stepped bar as shown in Figure 4. It is manufactured from a single piece of any homogeneous, isotropic material that is free of residual stress. The design of the Reference Material is parametric with a cantilever of thickness T, width 10T, and length 40T, with at one end an enlarged portion of thickness 5T and length 20T, giving an overall length of 60T. The manufacturing tolerances should be 0.05mm. The fillet radius at the transition between the cantilever and the enlarged portion should be as small as it is practical to manufacture, in order to minimise the effect of the strain distribution in the cantilever.

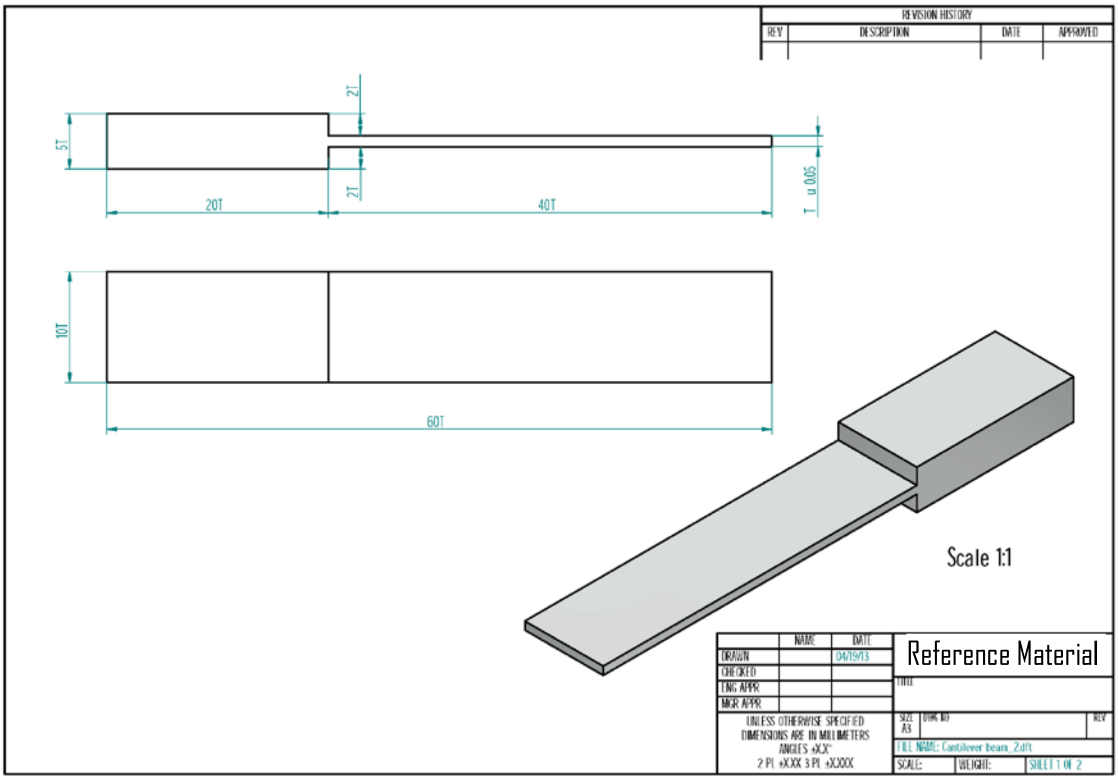

Figure 4: Drawing and three-dimensional rendering of the Reference Material. The thick section is intended to be rigidly clamped to a large, immovable body.

The design is scalable, and the dimensions and materials should be chosen so that deformations are comparable to those it is expected to measure with the calibrated measurement system; in addition, the Reference Material should occupy the majority of the field of view of the optical arrangement set-up for the planned experiments.

Appendix B: Normalized mode shapes

Figure 5 shows the analytic resonant bending mode shapes for a cantilever with a clamped and a free end. Note that the maximum deflection occurs at the tip for all modes.

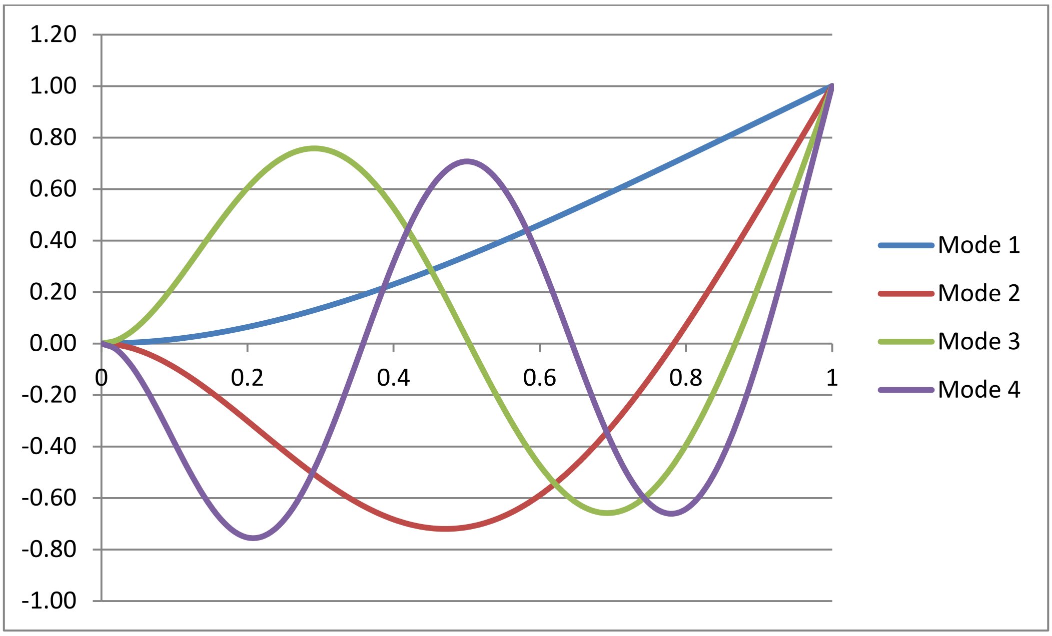

Figure 5: First four normalized bending modes of a cantilever clamped at one end.

Appendix C: Measurement uncertainty

There are several contributions to the uncertainty in the reference values obtained from the analytical expression:

- Uncertainty from the geometry
- Uncertainty from the idealization of the analytical expression
- Uncertainty from the identification of homologous points from the experiment and the RM surface.
- Uncertainty from the tip deflection measurement *u(δ)*

All quantities of interest for calibration are proportional to the tip deflection δ and can be expressed in reduced coordinates $\xi=x/L$. Then the propagation of uncertainties is used to determine the uncertainty of the quantities which leads to the following formulae:

The modal shape of the cantilever in bending mode k and its uncertainty are

$$w_k(\xi)=\frac{\delta}{2}\{\cosh\lambda_k(1-\xi)+\cos\lambda_k(1-\xi)-\phi_k(\sinh\lambda_k(1-\xi)+\sin\lambda_k(1-\xi))\}$$
$$u^2(w_k)=\frac{u^2(\delta)}{\delta^2}w_k{}^2+\left(\frac{\delta}{2}\lambda_k\right)^2\{\sinh\lambda_k(1-\xi)-\sin\lambda_k(1-\xi)-\phi_k(\cosh\lambda_k(1-\xi)+\cos\lambda_k(1-\xi))\}^2u^2(\xi) \tag{C1}$$

The strain along the cantilever in bending mode k, ε_{xk} and its uncertainty are

$$\varepsilon_{xk}(\xi)=\delta\frac{T}{4}\left(\frac{\lambda_k}{L}\right)^2\{\cosh\lambda_k(1-\xi)-\cos\lambda_k(1-\xi)-\phi_k(\sinh\lambda_k(1-\xi)-\sin\lambda_k(1-\xi))\}$$
$$u^2(\varepsilon_{xk})=\left[\frac{u^2(\delta)}{\delta^2}+\frac{u^2(T)}{T^2}+4\frac{u^2(L)}{L^2}\right]\varepsilon_{xk}{}^2 \quad + \tag{C2}$$
$$+\quad\left[\delta\frac{T}{4}\left(\frac{\lambda_i}{L}\right)^2\lambda_k\right]^2\{\sinh\lambda_k(1-\xi)+\sin\lambda_k(1-\xi)-\phi_k(\cosh\lambda_k(1-\xi)-\cos\lambda_k(1-\xi))\}^2u^2(\xi)$$

The static deflection and its uncertainty are

$$w(\xi)=\frac{\delta}{2}\left(3\xi^2-\ \xi^3\right)$$
$$u^2(w)=\frac{u^2(\delta)}{\delta^2}w^2\quad+\quad\left(3\frac{\delta}{2}\right)^2\left(2\xi-\ \xi^2\right)^2u^2(\xi) \tag{C3}$$

The direct strain along the cantilever, ε_x , and its uncertainty are

$$\varepsilon_x(\xi)=\delta\frac{3T}{2L^2}(1-\xi)$$
$$u^2(\varepsilon_x)=\left[\frac{u^2(\delta)}{\delta^2}+\frac{u^2(T)}{T^2}+4\frac{u^2(L)}{L^2}\right]\varepsilon_x{}^2\quad+\quad\left(\delta\frac{3}{80L}\right)^2u^2(\xi) \tag{C4}$$

The uncertainty of the tip deflection measurement is given by the calibration uncertainty of the transducer and the relative positioning uncertainty:

$$u^2(\delta) = u_{cal}{}^2(transducer) + \left(\frac{\partial q}{\partial \xi}\right)\Bigg|_{\xi=1}^{2} u^2(\xi_{pos}) \tag{C5}$$

where q is the dynamic, w_k, or static, w, deflection for which

$$\frac{\partial w_k}{\partial \xi} = \frac{\delta}{2}\lambda_k \left\{\sinh\lambda_k(0) - \sin\lambda_k(0) - \phi_k(\cosh\lambda_k(0) + \cos\lambda_k(0))\right\} = -\delta\,\lambda_k\,\phi_k$$
$$\frac{\partial w}{\partial \xi} = 3\frac{\delta}{2}\left(2 - 1^2\right) = \frac{3}{2}\delta \tag{C6}$$

which leads to the expressions

$$\frac{u^2(\delta)}{\delta^2} = \frac{u_{cal}{}^2(transducer)}{\delta^2} + (\lambda_k\phi_k)^2\, u^2(\xi_{pos})$$
$$\frac{u^2(\delta)}{\delta^2} = \frac{u_{cal}{}^2(transducer)}{\delta^2} + 1.5^2\, u^2(\xi_{pos}) \tag{C7}$$

for the dynamic and static cases, respectively.

The geometric uncertainties *u(T)* and *u(L)* have been obtained from the measurement of the geometry.

The uncertainty of the identification of homologous points from the experiment and the RM surface is represented by $u(\xi)$. A rough estimate would be given by one pixel of the detector. If the length L of the cantilever is images into K points, then the uncertainty would be $u(\xi)=1/K$. Of course, the uncertainty contributions due to the edge identification, image distortion, and perspective should be included.

The uncertainty from the idealization of the analytical expression can be assessed by a comparison of the analytical deflection and an FE calculation.

All expressions above for the uncertainty of measurand *m* can be cast into the form

$$u^2(m) = \left[\frac{u_{cal}{}^2(transducer)}{\delta^2} + F^2 \right] m^2 \quad + \quad \delta^2 G^2 u^2(\xi) \tag{C8}$$

The respective functions F^2 and G^2 are given in **Table 2**.

Table 2: Uncertainty contributions.

m	F^2	G^2
dynamic measurement in resonant mode *k*		
w_k	$(\lambda_k \phi_k)^2 \, u^2(\xi_{pos})$	$\frac{\lambda_k{}^2}{4}\{\sinh\lambda_k(1-\xi)-\sin\lambda_k(1-\xi)-\phi_k(\cosh\lambda_k(1-\xi)+\cos\lambda_k(1-\xi))\}^2$
ε_{xk}	$(\lambda_k \sigma_k)^2 \, u^2(\xi_{pos}) + \frac{u^2(T)}{T^2} + 4\frac{u^2(L)}{L^2}$	$\left(\frac{\lambda_k{}^3}{160L}\right)^2 \{\sinh\lambda_k(1-\xi)+\sin\lambda_k(1-\xi)-\phi_k(\cosh\lambda_k(1-\xi)-\cos\lambda_k(1-\xi))\}^2$
static measurement		
w	$1.5^2 \, u^2(\xi_{pos})$	$1.5^2(2\xi - \xi^2)^2$
ε_x	$1.5^2 \, u^2(\xi_{pos}) + \frac{u^2(T)}{T^2} + 4\frac{u^2(L)}{L^2}$	$\left(\frac{3}{80L}\right)^2$